Ch.4 Dup. for your files

THE LINK BETWEEN THE SACHS AND O(3) THEORIES OF ELECTRODYNAMICS

M. W. EVANS

CONTENTS

I. INTRODUCTION

In this volume, Sachs [I] has demonstrated, using irreducible representations of the Einstein group, that the electromagnetic field can propagate only in curved spacetime, implying that the electromagnetic field tensor can exist only when there is a nonvanishing curvature tensor $\kappa_{\mu\nu}$. Using this theory, Sachs has shown that the structure of electromagnetic theory is in general non-Abelian. This is the same overall conclusion as reached in O(3) electrodynamics [2], developed in the second chapter of this volume. In this short review, the features common to Sachs and O(3) electrodynamics are developed. The $\boldsymbol{B}^{(3)}$ field of O(3) electrodynamics is extracted from the quatemion-valued $B^{\mu\nu}$ equivalent in the Sachs theory; the most general form of the vector potential is considered in both theories, the covariant derivatives are compared in both theories, and the possibility of extracting energy from the vacuum is considered in both theories.

Modern Nonlinear Optics, Pun 2. Second Edition, Advances in Chemical Physics, Volume 119,
Edited by Myron W. Evans. Series Editors I. Prigogine and Stuart A. Rice.
ISBN O-471-38931-5

II. THE NON-ABELIAN STRUCTURE OF THE FIELD TENSOR

The non-Abelian component of the field tensor is defined through a metric q^μ that is a set of four quatemion-valued components of a 4-vector, a 4-vector each of whose components can be represented by a 2 x 2 matrix. In condensed notation:

$$q^\mu = (q^{\mu 0}, q^{\mu 1}, q^{\mu 2}, q^{\mu 3}) \tag{1}$$

and the total number of components of q^μ is 16. The covariant and second covariant derivatives of q^μ vanish [I] and the line element is given by

$$ds = q^\mu(x) dx_\mu \tag{2}$$

which, in special relativity (flat spacetime), reduces to

$$ds = \sigma^\mu dx_\mu \tag{3}$$

where σ^μ is a 4-vector made up of **Pauli** matrices:

$$\sigma^\mu = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \tag{4}$$

In the limit of special relativity

$$q^\mu q^{\nu *} - q^\nu q^{\mu *} \rightarrow \sigma^\mu \sigma^\nu - \sigma^\nu \sigma^\mu \tag{5}$$

where * denotes reversing the time component of the quaternion-valued q^μ. The most general form of the non-Abelian part of the electromagnetic field tensor in conformally curved spacetime is 1

$$F^{\mu\nu} = \frac{1}{8} QR(q^\mu q^{\nu *} - q^\nu q^{\mu *}) \tag{6}$$

To consider magnetic flux density components of $F^{\mu\nu}$, Q must have the units of weber and R, the scalar curvature, must have units of inverse square meters. In the flat spacetime limit, $R = 0$, so it is clear that the non-Abelian part of the field tensor, Eq. (6), vanishes in special relativity. The complete field tensor $F^{\mu\nu}$ vanishes [1] in flat spacetime because the curvature tensor vanishes. These considerations refute the Maxwell-Heaviside theory, which is developed in flat spacetime, and show that O(3) electrodynamics is a theory of conformally curved spacetime. Most generally, the Sachs theory is a closed field theory that, in principle, unifies ail four **fields**: gravitational, electromagnetic, weak, and strong.

There exist generally covariant four-valued 4-vectors that are components of q^{μ}, and these can be used to construct the basic structure of O(3) electrodynamics in terms of single-valued components of the quaternion-valued metric q^{μ}. Therefore, the Sachs theory can be reduced to O(3) electrodynamics, which is a Yang-Mills theory [3,4]. The empirical evidence available for both the Sachs and O(3) theories is summarized in this review, and discussed more extensively in the individual reviews by Sachs [1] and Evans [2]. In other words, empirical evidence is given of the instances where the Maxwell-Heaviside theory fails and where the Sachs and O(3) electrodynamics succeed in describing empirical data from various sources. The fusion of the O(3) and Sachs theories provides proof that the $8^{(3)}$ field [2] is a physical field of curved spacetime, which vanishes in flat spacetime (Maxwell-Heaviside theory [2]).

In Eq. (5), the product $q^{\mu}q^{\nu *}$ is quaternion-valued and noncommutative, but not antisymmetric in the indices μ and v. The $\boldsymbol{B}^{(3)}$ field and structure of O(3) electrodynamics must be found from a special case of Eq. (5) showing that O(3) electrodynamics is a Yang-Mills theory and also a theory of general relativity [l]. The important conclusion reached is that Yang-Mills theories can be derived from the irreducible representations of the Einstein group. This result is consistent with the fact that all theories of physics must be theories of general relativity in principle. From Eq. (1), it is possible to write four-valued, generally covariant, components such as

$$q_X = \left(q_X^0, q_X^1, q_X^2, q_X^3\right) \tag{7}$$

which, in the limit of special relativity, reduces to

$$\sigma_x = (0, \sigma_x, 0, 0) \tag{8}$$

Similarly, one can write

$$q_Y = \left(q_Y^0, q_Y^1, q_Y^2, q_Y^3\right) \to (0, 0, \sigma_Y, 0) \tag{9}$$

and use the property

$$q_X q_Y^* - q_Y q_X^* \to \sigma_X \sigma_Y - \sigma_Y \sigma_X \tag{10}$$

in the limit of special relativity. The only possibility from Eqs. (7) and (9) is that

$$\begin{gathered} q_X^1 q_Y^{2*} \quad q_Y^2 q_X^{1*} = 2iq_Z^3 \\ \downarrow \\ \sigma_X \sigma_Y \quad \sigma_Y \sigma_X = 2i\sigma_Z \end{gathered} \tag{11}$$

where q_x^1 is single valued. In a 2 x 2 matrix representation, this is

$$q_X^1 = \begin{bmatrix} 0 & q_X^1 \\ q_X^1 & 0 \end{bmatrix} \to \sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{12}$$

Similarly

$$q_Y^{2*} = \begin{bmatrix} 0 & -iq_Y^2 \\ iq_Y^2 & 0 \end{bmatrix} \to \sigma_Y = \begin{bmatrix} o & -i \\ i & 0 \end{bmatrix} \tag{13}$$

$$q_Y^3 = \begin{bmatrix} q_Z^3 & 0 \\ 0 & -q_Z^3 \end{bmatrix} \to \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{14}$$

Therefore, there exist cyclic relations with O(3) symmetry

$$\begin{aligned} q_X^1 q_Y^{2*} - q_Y^2 q_X^{1*} &= 2iq_Z^3 \\ q_Y^2 q_Z^{3*} - q_Z^3 q_Y^{2*} &= 2iq_X^1 \\ q_Z^3 q_X^{1*} - q_X^1 q_Z^{3*} &= 2iq_Y^2 \end{aligned} \tag{15}$$

and the structure of O(3) electrodynamics [2] begins to emerge. If the space basis is represented by the complex circular ((1),(2),(3)) then Eqs. (15) become

$$\begin{aligned} q_X^{(1)} q_Y^{(2)*} - q_Y^{(2)} q_X^{(1)*} &= 2iq_Z^{(3)} \\ q_Y^{(2)} q_Z^{(3)*} - q_Z^{(3)} q_Y^{(2)*} &= 2iq_X^{(1)} \\ q_Z^{(3)} q_X^{(1)*} - q_X^{(1)} q_Z^{(3)*} &= 2iq_Y^{(2)} \end{aligned} \tag{16}$$

These are cyclic relations between single-valued metric field components in the non-Abelian part [Eq. (6)] of the quaternion-valued $F^{\mu\nu}$. Equation (16) can be put in vector form

$$\begin{aligned} \boldsymbol{q}^{(1)} \times \boldsymbol{q}^{(2)} &= i\boldsymbol{q}^{(3)*} \\ \boldsymbol{q}^{(2)} \times \boldsymbol{q}^{(3)} &= i\boldsymbol{q}^{(1)*} \\ \boldsymbol{q}^{(3)} \times \boldsymbol{q}^{(1)} &= i\boldsymbol{q}^{(2)*} \end{aligned} \tag{17}$$

where the asterisk denotes ordinary complex conjugation in Eq. (17) and quaternion conjugation in Eq. (16).

Equation (17) contains vector-valued metric fields in the complex basis ((1),(2),(3)) [2]. Specifically, in O(3) electrodynamics, which is based on the

existence of two circularly polarized components of electromagnetic radiation [2]

$$q^{(1)} = \frac{1}{\sqrt{2}} (ii + j) \exp(i\phi) \tag{18}$$

$$q^{(2)} = \&(-ii + j) \exp(i\phi) \tag{19}$$

giving

$$q^{(3)*} = k \tag{20}$$

and

$$B^{(3)} = \frac{1}{8} QRq^{(3)} \tag{21}$$

Therefore, the $B^{(3)}$ field [2] is proved from a particular choice of metric using the irreducible representations of the Einstein group [1]. It can be seen from Eq. (21) that the $B^{(3)}$ field is the vector-valued metric field $q^{(3)}$ within a factor $\frac{1}{8}$ QR. This result proves that $B^{(3)}$ vanishes in flat spacetime, because $R = 0$ in flat spacetime. If we write

$$B^{(3)} = \frac{1}{8} QR \tag{22}$$

then Eq. (17) becomes the B cyclic theorem [2] of O(3) electrodynamics:

$$B^{(1)} \times B^{(2)} = iB^{(0)}B^{(3)*} \tag{23}$$

Since O(3) electrodynamics is a Yang-Mills theory [3,4], we can write

$$q = q^{(1)}i + q^{(2)}j + q^{(3)}k \tag{24}$$

from which it follows [5] that

$$D^{\mu}(D_{\mu}q) = 0; \quad D_{\mu}q = 0 \tag{25}$$

Thus the first and second covariant derivatives vanish [1].

The Sachs theory [I] is able to describe parity violation and spin-spin interactions from first principles [6] on a classical level; it can also explain

several problems of neutrino physics, and the Pauli exclusion principle can be derived from it classically. The quaternion form of the theory [1], which is the basis of this review chapter, predicts small but nonzero masses for the neutrino and photon; describes the Planck spectrum of blackbody radiation classically; describes the Lamb shifts in the hydrogen atom with precision equivalent to quantum electrodynamics, but without renormalization of infinities; proposes grounds for charge quantization; predicts the lifetime of the muon state; describes electron-muon mass splitting; predicts physical longitudinal and timelike photons and fields; and has bult-in ***P*, *C*,** and ***T*** violation.

To this list can now be added the advantages of O(3) over U(1) electrodynamics, advantages that are described in the review by Evans in Part 2 of this three-volume set and by Evans, Jeffers, and Vigier in Part 3. In summary, by interlocking the Sachs and O(3) theories, it becomes apparent that the advantages of O(3) over U(1) are symptomatic of the fact that the electromagnetic field vanishes in flat spacetime (special relativity), if the irreducible representations of the Einstein group are used.

III. THE COVARIANT DERIVATIVE

The covariant derivative in the Sachs theory [1] is defined by the spin-affine connection:

$$D^\rho = \partial^\rho + \Omega^\rho \tag{26}$$

where

$$\Omega_\mu = \frac{1}{4}(\partial_\mu q^\rho + \Gamma^\rho_{\tau\mu} q^\tau) q^*_\rho \tag{27}$$

and where $\Gamma^\rho_{\tau\mu}$ is the Christoffel symbol. The latter can be defined through the reducible metrics $g_{\mu\nu}$ as follows [1]:

$$\Gamma^\rho_{\mu\alpha} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\lambda\alpha} + \partial_\alpha g_{\mu\lambda} - \partial_\lambda g_{\alpha\mu}) \tag{28}$$

In O(3) electrodynamics, the covariant derivative on the classical level is defined by

$$D_\mu = \partial_\mu - igA_\mu = \partial_\mu - ig\mathrm{M}^a A^a_\mu \tag{29}$$

where $\mathbf{M}^a$ are rotation generators [2] of the O(3) group, and where a is an internal index of Yang-Mills theory. The complete vector potential in O(3) electrodynamics is defined by

$$A = A^{(1)} e^{(2)} + A^{(2)} e^{(1)} + A^{(3)} e^{(3)} \tag{30}$$

where $e^{(1)}$, $e^{(2)}$, $e^{(3)}$ are unit vectors of the complex circular basis ((l),(2),(3)) [2]. If we restrict our discussion to plane waves, then the vector potential is

$$A^{(1)} = \frac{A^{(0)}}{\sqrt{2}} \text{ (ii i-j) } \exp\ (i\phi) \tag{31}$$

where ϕ is the electromagnetic phase. Therefore, there are O(3) electrodynamics components such as

$$A_X^{(1)} = \frac{iA^{(0)}}{\sqrt{2}} e^{(i\phi)}; \quad A_Y^{(1)} = \frac{A^{(0)}}{\sqrt{2}} e^{(i\phi)} \tag{32}$$

In order to reduce the covariant derivative in the Sachs theory to the O(3) covariant derivative, the following classical equation must hold:

$$-igA_\mu = \frac{1}{4}(D_\mu q^\rho)q_\rho^* \tag{33}$$

This equation can be examined component by component, giving relations such as

$$-igA_X^{(1)} = -\frac{1}{4}(D_X q_Y^{(1)})A_Y^{(1)} \tag{34}$$

where we have used

$$q_Y^{(1)} = -iq_X^{(1)} \tag{35}$$

Using [2]

$$g = \frac{\kappa}{A^{(0)}} \tag{36}$$

we obtain

$$i\kappa q_X^{(1)} = \frac{1}{4}(D_X q_Y^{(1)})q_Y^{(1)} = -\frac{i}{4}(D_X q_Y^{(1)})q_X^{(1)} \tag{37}$$

so that the wavenumber κ is defined by

$$\kappa = -\frac{1}{4} D_X q_Y^{(1)} \tag{38}$$

Therefore, we can write

$$D_X q_Y^{(1)} = D_1 q^{1(1)} = \partial_1 q^{1(1)} + \Gamma^1_{\lambda 1} q^{\lambda(1)} \tag{39}$$

and the wavenumber becomes the following sum:

$$\kappa = -\frac{1}{4}\left(\Gamma^1_{11}q^{1(1)} + \Gamma^1_{21}q^{2(1)}\right) \tag{40}$$

Using the identities

$$q^{1(1)} = q_X^{(1)} = \frac{i}{\sqrt{2}}e^{i\phi} \tag{41}$$

$$q^{2(1)} = q_Y^{(1)} = \frac{1}{\sqrt{2}}e^{i\phi} \tag{42}$$

the wavenumber becomes

$$\kappa = -\frac{1}{4}\left(\frac{i\Gamma^1_{11}}{\sqrt{2}}e^{i\phi} + \frac{\Gamma^1_{21}}{\sqrt{2}}e^{i\phi}\right) \tag{43}$$

Introducing the definition (28) of the Christoffel symbol, it is possible to write

$$\begin{aligned}\Gamma^1_{11} &= \frac{1}{2}g^{1\lambda}(\partial_1 g_{\lambda 1} + \partial_1 g_{1\lambda} - \partial_1 g_{11})\\ &= \frac{1}{2}g^{13}\partial_Z g_{11} + \cdots\end{aligned} \tag{44}$$

so that

$$\kappa = -\frac{i}{8\sqrt{2}}g^{13}\partial_Z g_{11}e^{i\phi} + \ldots \tag{45}$$

This equation is satisfied by the following choice of metric:

$$g_{11} = \frac{1}{2};\ g^{13} = -8\sqrt{2}\,e^{-i\phi} \tag{46}$$

Similarly

$$\begin{aligned}\Gamma^1_{21} &= \frac{1}{2}g^{1\lambda}(\partial_2 g_{\lambda 1} + \partial_1 g_{2\lambda} - \partial_\lambda g_{12})\\ &= \frac{1}{2}g^{13}\partial_Z g_{12} + \cdots\end{aligned} \tag{47}$$

so that the wavenumber can be expressed as

$$\kappa = \frac{i\kappa}{8\sqrt{2}}g^{13}g_{12}e^{i\phi} \tag{48}$$

an equation that is satisfied by the following choice of metric:

$$g_{12} = \frac{i}{2}; \quad g^{13} = -8\sqrt{2}\, e^{-i\phi} \tag{49}$$

Therefore, it is always possible to write the covariant derivative of the Sachs theory as an O(3) covariant derivative of O(3) electrodynamics. Both types of covariant derivative are considered on the classical level.

IV. ENERGY FROM THE VACUUM

The energy density in curved spacetime is given in the Sachs theory by the quaternion-valued expression

$$En_d = A^{\mu} j_{\mu}^{*} \tag{50}$$

where A^{μ} is the quaternion-valued vector potential and J_{μ}^{*} is the quaternion-valued 4-current as given by Sachs [I]. Equation (50) is an elegant and deeply meaningful expression of the fact that electromagnetic energy density is available from curved spacetime under all conditions; the distinction between field and matter is lost, and the concepts of "point charge" and "point mass" are not present in the theory, as these two latter concepts represent infinities of the closed-field theory developed by Sachs [1] from the irreducible representations of the Einstein group. The accuracy of expression (50) has been tested [1] to the precision of the Lamb shifts in the hydrogen atom without using renormalization of infinities. The Lamb shifts can therefore be viewed as the results of electromagnetic energy from curved spacetime.

Equation (50) is geometrically a scalar and algebraically quaternion-valued equation [1], and it is convenient to develop it using the identity [1]

$$q_{\gamma} q^{\kappa *} + q^{\kappa} q_{\gamma}^{*} = 2\sigma_0 \delta_{\gamma}^{\kappa} \tag{51}$$

with the indices defined as

$$\gamma = \kappa = \mu \tag{52}$$

to obtain

$$q^{\mu} q_{\mu}^{*} = \sigma_0 \delta^{\mu} \tag{53}$$

Using summation over repeated indices on the right-hand side, we obtain the following result:

$$q^{\mu} q_{\mu}^{*} = 4\sigma_0 \tag{54}$$

In the limit of flat spacetime

$$q^{\mu}q_{\mu}^{*} \rightarrow \sigma^{\mu}\sigma_{\mu} = 4\sigma_{0} \tag{55}$$

where the right-hand side is again a scalar invariant geometrically and a quaternion algebraically.

Therefore, the energy density (50) assumes the simple form

$$A\mu J_{\mu}^{*} = 4A_{0}J_{0}^{*}\sigma_{0} \tag{56}$$

A_0 and J_0^* are magnitudes of A^{μ} and J_{μ}^*. In flat spacetime, this electromagnetic energy density vanishes because the curvature tensor vanishes. Therefore, in the Maxwell-Heaviside theory, there is no electromagnetic energy density from the vacuum and the field does not propagate through flat spacetime (the vacuum of the Maxwell-Heaviside theory) because of the absence of curvature. The $\boldsymbol{B}^{(3)}$ field depends on the scalar curvature R in Eq. (21), and so the $\boldsymbol{B}^{(3)}$ field and O(3) electrodynamics are theories of conformally curved spacetime. To maximize the electromagnetic energy density, the curvature has to be maximized, and the maximization of curvature may be the result of the presence of a gravitating object. In general, wherever there is curvature, there is electromagnetic energy that may be extracted from curved spacetime using a suitable device such as a dipole [7].

Therefore, we conclude that electromagnetic energy density exists in curved spacetime under all conditions, and devices can be constructed [8] to extract this energy density.

The quaternion-valued vector potential A^{μ} and the 4-current J_{μ}^* both depend directly on the curvature tensor. The electromagnetic field tensor in the Sachs theory has the form

$$F_{\mu\nu} = \partial_{\mu}A_{\nu}^{*} - \partial_{\nu}A_{\mu}^{*} + \frac{1}{8}QR(q_{\mu}q_{\nu}^{*} - q_{\nu}q_{\mu}^{*}) \tag{57}$$

where the quaternion-valued vector potential is defined as

$$\text{A,} = \frac{Q}{4}q_{\gamma}^{*}\int(\kappa_{\rho\lambda}q^{\lambda} + q^{\lambda}\kappa_{\rho\lambda}^{+})\,dx^{\rho} \tag{58}$$

The most general form of the vector potential is therefore given by Eq. (58), and if there is no curvature, the vector potential vanishes.

Similarly, the 4-current J_{μ}^* depends directly on the curvature tensor $\kappa_{\rho\lambda}$ [1], and there can exist no 4-current in the Heaviside-Maxwell theory, so the 4-current cannot act as the source of the field. In the closed-field theory,

represented by the irreducible representations of the Einstein group [1], charge and current are manifestations of curved spacetime, and can be regarded as the results of the field. This is the viewpoint of Faraday and Maxwell rather than that of Lorentz. It follows that there can exist a vacuum 4-current in general relativity, and the implications of such a current are developed by Lehnert [9]. The vacuum 4-current also exists in O(3) electrodynamics, as demonstrated by Evans and others [2,9]. The concept of vacuum 4-current is missing from the flat spacetime of Maxwell-Heaviside theory.

In curved spacetime, both the electromagnetic and curvature 4-tensors may have longitudinal as well as transverse components in general and the electromagnetic field is always accompanied by a source, the 4-current J_μ^*. In the Maxwell-Heaviside theory, the field is assumed incorrectly to propagate through flat spacetime without a source, a violation of both causality and general relativity. As shown in several reviews in this three-volume set, Maxwell-Heaviside theory and its quantized equivalent appear to work well only under certain incorrect assumptions, and quantum electrodynamics is not a physical theory because, as pointed out by Dirac and many others, it contains infinities. Sachs [1] has also considered and removed the infinite self-energy of the electron by a consideration of general relativity.

The O(3) electrodynamics developed by Evans [2], and its homomorph, the SU(2) electrodynamics of Barrett [10], are substructures of the Sachs theory dependent on a particular choice of metric. Both O(3) and SU(2) electrodynamics are Yang-Mills structures with a Wu-Yang phase factor, as discussed by Evans and others [2,9]. Using the choice of metric (17), the electromagnetic energy density present in the O(3) curved spacetime is given by the product

$$En_d = \boldsymbol{A} \cdot \boldsymbol{j} \tag{59}$$

where the vector potential and 4-current are defined in the ((l),(2),(3)) basis in terms of the unit vectors similar to those in Eq. (2), and as described elsewhere in this three-volume set [2]. The extraction of electromagnetic energy density from the vacuum is also possible in the Lehnert electrodynamics as described in his review in the first chapter of this volume (i.e., here, in Part 2 of this three-volue set). The only case where extraction of such energy is not possible is that of the Maxwell-Heaviside theory, where there is no curvature.

The most obvious manifestation of energy from curved spacetime is gravitation, and the unification of gravitation and electromagnetism by Sachs [1] shows that electromagnetic energy emanates under all circumstances from spacetime curvature. This principle has been tested to the precision of the Lamb shifts of H as discussed already. This conclusion means that the electromagnetic field does not emanate from a “point charge,” which in general relativity can be present only when the curvature becomes infinite. The concept of “point

charge" is therefore unphysical, and this is the basic reason for the infinite electron self-energy in the Maxwell-Heaviside theory and the infinities of quantum electrodynamics, a theory rejected by Einstein, Dirac, and several other leading scientists of the twentieth century. The electromagnetic energy density inherent in curved spacetime depends on curvature as represented by the curvature tensor discussed in the next section. In the Einstein field equation of general relativity, which comes from the reducible representations of the Einstein group [1], the canonical energy momentum tensor of gravitation depends on the Einstein curvature tensor.

Sachs [1] has succeeded in unifying the gravitational and electromagnetic fields so that both share attributes. For example, both fields are non-Abelian under all conditions, and both fields are their own sources. The gravitational field carries energy that is equivalent to mass [1 1], and so is itself a source of gravitation. Similarly, the electromagnetic field carries energy that is equivalent to a 4-current, and so is itself a source of electromagnetism. These concepts are missing entirely from the Maxwell-Heaviside theory, but are present in O(3) electrodynamics, as discussed elsewhere [2,10]. The Sachs theory cannot be reduced to the Maxwell-Heaviside theory, but can be reduced, as discussed already, to O(3) electrodynamics. The fundamental reason for this is that special relativity is an asymptotic limit of general relativity, but one that is never reached precisely [1]. So the Poincaré group of special relativity is not a subgroup of the Einstein group of general relativity.

In standard Maxwell-Heaviside theory, the electromagnetic field is thought of as propagating in a source-free region in flat spacetime where there is no curvature. If, however, there is no curvature, the electromagnetic field vanishes in the Sachs theory [1], which is a direct result of using irreducible representations of the Einstein group of standard general relativity. The empirical evidence for the Sachs theory has been reviewed in this chapter already, and this empirical evidence refutes the Maxwell-Heaviside theory. In general relativity [1], if there is mass or charge anywhere in the universe, then the whole of spacetime is curved, and all the laws of physics must be written in curved spacetime, including, of course, the laws of electrodynamics. Seen in this light, the O(3) electrodynamics of Evans [2] and the homomorphic SU(2) electrodynamics of Barrett [12] are written correctly in conformally curved spacetime, and are particular cases of Einstein's general relativity as developed by Sachs [1]. Flat spacetime as the description of the vacuum is valid only when the whole universe is empty.

From everyday experience, it is possible to extract gravitational energy from curved spacetime on the surface of the earth. The extraction of electromagnetic energy must be possible if the extraction of gravitational energy is possible, and the electromagnetic field influences the gravitational field and vice versa. The field equations derived by Sachs [1] for electromagnetism are complicated, but

can be reduced to the equations of O(3) electrodynamics by a given choice of metric. The literature discusses the various ways of solving the equations of O(3) electrodynamics [2,10], analytically, or using computation. In principle, the Sachs equations are solvable by computation for any given experiment, and such a solution would show the reciprocal influence between the electromagnetic and gravitational fields, leading to significant findings.

The ability of extracting electromagnetic energy density from the vacuum depends on the use of a device such as a dipole, and this dipole can be as simple as battery terminals, as discussed by Bearden [13] The principle involved in this device is that electromagnetic energy density $A^{\mu}\ J_{\mu}^{*}$ exists in general relativity under all circumstances, and electromagnetic 4-currents and 4-potentials emanate form spacetime curvature. Therefore, the current in the battery is not driven by the positive and negative terminals, but is a manifestation of energy from curved spacetime, just as the hydrogen Lamb shift is another such manifestation. A battery runs down because the chemical energy needed to form the dipole dissipates.

In principle, therefore, the electromagnetic energy density in Eq. (50) is always available whenever there is spacetime curvature; in other words, it is always available because there is always spacetime curvature.

V. THE CURVATURE TENSOR

The curvature tensor is defined in terms of covariant derivatives of the spin–affine connections Ω_{ρ}, and according to Section (III), has its equivalent in O(3) electrodynamics.

The curvature tensor is

$$\begin{aligned}\kappa_{\rho\lambda} = -\kappa_{\lambda\rho} &= \Omega_{\rho:\lambda} - \Omega_{\lambda:\rho} \\ &= \partial_{\lambda}\Omega_{\rho} - \partial_{\rho}\Omega_{\lambda} + \Omega_{\lambda}\Omega_{\rho} - \Omega_{\rho}\Omega_{\lambda}\end{aligned} \tag{60}$$

and obeys the Jacobi identity

$$D_{\gamma}\kappa_{\rho\lambda} + D_{\rho}\kappa_{\lambda\gamma} + D_{\lambda}\kappa_{\gamma\rho} \equiv 0 \tag{61}$$

which can be written as

$$D_{\mu}\tilde{\kappa}^{\mu\nu} \equiv 0 \tag{62}$$

where

$$\tilde{\kappa}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\kappa_{\rho\sigma} \tag{63}$$

is the dual of $\kappa_{\rho\sigma}$.

Equation (4) has the form of the homogeneous field equation of O(3) electrodynamics [2,10]. If we now define

$$\begin{aligned}\kappa^{\rho\lambda} &= \Omega^{\rho;\lambda} - \Omega^{\lambda;\rho} \\ &= (\partial^\lambda + \Omega^\lambda)\Omega^\rho - (\partial^\rho + \Omega^\rho)\Omega^\lambda \end{aligned} \tag{64}$$

then

$$\begin{aligned}D_\rho\kappa^{\rho\lambda} &= (\partial_\rho + \Omega_\rho)((\partial^\lambda + \Omega^\lambda)\Omega^\rho - (\partial^\rho + \Omega^\rho)\Omega^\lambda) \\ &\equiv L^\lambda \neq 0\end{aligned} \tag{65}$$

has the form of the inhomogeneous field equation of O(3) electrodynamics with a nonzero source term L^λ in curved spacetime.

The curvature tensor can be written as a commutator of covariant derivatives

$$\begin{aligned}\kappa_{\mu\nu} &= -\kappa_{\nu\mu} = -[D_\mu, D_\nu] = -[\partial_\mu + \Omega_\mu, \partial_\nu + \Omega_\nu] \\ &= \Omega_{\mu;\nu} - \Omega_{\nu;\mu}\end{aligned} \tag{66}$$

and is the result of a closed loop, or holonomy, in curved spacetime. This is the way in which a curvature tensor is also derived in general gauge field theory on the classical level [11]. If a field ϕ is introduced such that

$$\phi'(x) = S\phi(x) \tag{67}$$

under a gauge transformation, it follows that

$$\delta\phi = \Omega_\mu dx^\mu \phi \tag{68}$$

and that

$$\partial_\mu\phi' = S(\partial_\mu\phi) + (\partial_\mu S)\phi \tag{69}$$

The expression equivalent to Eq. (68) in general gauge field theory is [11]

$$\delta\psi = igM^a A^a_\mu dx^\mu \psi \tag{70}$$

where M^a are group rotation generators and A^a_μ are vector potential components with internal group indices a. Under a gauge transformation

$$(\partial_\mu + \Omega'_\mu)\phi' = S(\partial_\mu + \Omega_\mu)\phi \tag{71}$$

leading to the expression

$$\Omega'_\mu = S\Omega_\mu S^{-1} - (\partial_\mu S)S^{-1} \tag{72}$$

The equivalent equation in general gauge field theory is

$$A'_\mu = SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1} \tag{73}$$

Equations (72) and (73) show that the spin-affine connection Ω_μ and vector potential A_μ behave similarly under a gauge transformation. The relation between covariant derivatives has been developed in Section III.

VI. -GENERALLY COVARIANT 4-VECTORS

The most fundamental feature of O(3) electrodynamics is the existence of the $\boldsymbol{B}^{(3)}$ field [2], which is longitudinally directed along the axis of propagation, and which is defined in terms of the vector potential plane wave:

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} \tag{74}$$

From the irreducible representations of the Einstein group, there exist 4-vectors that are generally covariant and take the following form:

$$\begin{aligned} B_1^\mu &= (B_X^{(0)}, B_X^{(1)}, B_X^{(2)}, B_X^{(3)}) \\ B_2^\mu &= (B_Y^{(0)}, B_Y^{(1)}, B_Y^{(2)}, B_Y^{(3)}) \\ B_3^\mu &= (B_Z^{(0)}, B_Z^{(1)}, B_Z^{(2)}, B_Z^{(3)}) \end{aligned} \tag{75}$$

All these components exist in general, and the $\boldsymbol{B}^{(3)}$ field can be identified as the $B_Z^{(3)}$ component. In O(3) electrodynamics, these 4-vectors reduce to

$$\begin{aligned} B_1^\mu &= (0, B_X^{(1)}, B_X^{(2)}, 0) \\ B_2^\mu &= (0, B_Y^{(1)}, B_Y^{(2)}, 0) \\ B_3^\mu &= (B_Z^{(0)}, 0, 0, B_Z^{(3)}) \end{aligned} \tag{76}$$

so it can be concluded that O(3) electrodynamics is developed in a curved spacetime that is defined in such a way that

$$\boldsymbol{B}^{(3)} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{77}$$

In O(3) electrodynamics, there exist the cyclic relations (23), and we have seen that in general relativity, this cyclic relation can be derived using a particular choice of metric. In the special case of O(3) electrodynamics, the vector

$$B_3^\mu = (B_Z^{(0)}, B_Z^{(1)}, B_Z^{(2)}, B_Z^{(3)}) \tag{78}$$

reduces to

$$B_3^\mu = (B_Z^{(0)}, 0, 0, B_Z^{(3)}) \tag{79}$$

Similarly, there exists, in general, the 4-vector

$$A_3^\mu = (A_Z^{(0)}, A_Z^{(1)}, A_Z^{(2)}, A_Z^{(3)}) \tag{80}$$

which reduces in O(3) electrodynamics to

$$A_3^\mu = (A_Z^{(0)}, 0, 0, A_Z^{(3)}) \tag{81}$$

and that corresponds to generally covariant energy-momentum.

The curved spacetime 4-current is also generally covariant and has components such as

$$\begin{aligned} j_1^\mu &= (j_X^{(0)}, j_X^{(1)}, j_X^{(2)}, j_X^{(3)}) \\ j_2^\mu &= (j_Y^{(0)}, j_Y^{(1)}, j_Y^{(2)}, j_Y^{(3)}) \\ j_3^\mu &= (j_Z^{(0)}, j_Z^{(1)}, j_Z^{(2)}, j_Z^{(3)}) \end{aligned} \tag{82}$$

which, in O(3) electrodynamics, reduce to

$$\begin{aligned} j_1^\mu &= (0, j_X^{(1)}, j_X^{(2)}, 0) \\ j_2^\mu &= (0, j_Y^{(1)}, j_Y^{(2)}, 0) \\ j_3^\mu &= (j_Z^{(0)}, 0, 0, j_Z^{(3)}) \end{aligned} \tag{83}$$

The existence of a vacuum current such as this is indicated in O(3) electrodynamics by its inhomogeneous field equation

$$D_\mu G^{\mu\nu} = J^\nu \tag{84}$$

which is a Yang-Mills type of equation [2]. The concept of vacuum current was also introduced by Lehnert and is discussed in his review (first chapter in this volume; i.e., in Part 2).

The components of the antisymmetric field tensor in the Sachs theory [1] are

$$\begin{aligned} B^3 &= F^{21} = -F^{12} = (B_Z^{(0)}, B_Z^{(1)}, B_Z^{(2)}, B_Z^{(3)}) \\ B^1 &= F^{32} = -F^{23} = (B_X^{(0)}, B_X^{(1)}, B_X^{(2)}, B_X^{(3)}) \\ B^2 &= F^{13} = -F^{31} = (B_Y^{(0)}, B_Y^{(1)}, B_Y^{(2)}, B_Y^{(3)}) \\ E^1 &= F^{01} = -F^{10} = (E_X^{(0)}, E_X^{(1)}, E_X^{(2)}, E_X^{(3)}) \\ E^2 &= F^{02} = -F^{20} = (E_Y^{(0)}, E_Y^{(1)}, E_Y^{(2)}, E_Y^{(3)}) \\ E^3 &= F^{03} = -F^{30} = (E_Z^{(0)}, E_Z^{(1)}, E_Z^{(2)}, E_Z^{(3)}) \end{aligned} \tag{85}$$

each of which is a 4-vector that is generally covariant. For example

$$B_Z^{\mu} B_{\mu Z} = \text{invariant} \tag{86}$$

So, in general, in curved spacetime, there exist longitudinal and transverse components under all conditions. In O(3) electrodynamics, the upper indices ((l),(2),(3)) are defined by the unit vectors

$$\begin{aligned} e^{(1)} &= \frac{1}{\sqrt{2}}(i - ij) \\ e^{(2)} &= \frac{1}{\sqrt{2}}(i + ij) \\ e^{(3)} &= k \end{aligned} \tag{87}$$

which form the cyclically symmetric relation [2]

$$\begin{aligned} e^{(1)} \times e^{(2)} &= ie^{(3)*} \\ &\cdots \end{aligned} \tag{88}$$

where the asterisk in this case denotes complex conjugation. In addition, there is the time-like index (0). The field tensor components in O(3) electrodynamics are therefore, in general

$$\begin{aligned} F^{01} &= -F^{10} = (0, E_X^{(1)}, E_X^{(2)}, 0) \\ F^{02} &= -F^{20} = (0, E_Y^{(1)}, E_Y^{(2)}, 0) \\ F^{03} &= -F^{30} = (E_Z^{(0)}, 0, 0, E_Z^{(3)}) \\ F^{21} &= -F^{12} = (B_Z^{(3)}, 0, 0, B_Z^{(3)}) \\ F^{13} &= -F^{31} = (0, B_Y^{(1)}, B_Y^{(2)}, 0) \\ F^{32} &= -F^{23} = (0, B_X^{(1)}, B_X^{(2)}, 0) \end{aligned} \tag{89}$$

and the following invariants occur:

$$\begin{aligned} B_Y^{(1)} B_Y^{(2)} + B_Y^{(2)} B_Y^{(1)} &= B^{(0)2} \\ B_X^{(1)} B_X^{(2)} + B_X^{(2)} B_X^{(1)} &= B^{(0)2} \\ E_Y^{(1)} E_Y^{(2)} + E_Y^{(2)} E_Y^{(1)} &= E^{(0)2} \\ E_X^{(1)} E_X^{(2)} + E_X^{(2)} E_X^{(1)} &= E^{(0)2} \\ B_Z^{(0)2} - B_Z^{(3)2} = E_Z^{(0)2} - E_Z^{(3)2} &= 0 \end{aligned} \tag{90}$$

From general relativity, it can therefore be concluded that the $\boldsymbol{B}^{(3)}$ field must exist and that it is a physical magnetic flux density defined to the precision of the Lamb shift. It propagates through the vacuum with other components of the field tensor.

VII. SACHS THEORY IN THE FORM OF A GAUGE THEORY

The most general form of the vector potential can be obtained by writing the first **two** terms of Eq. (57) as

$$F_{\rho\gamma,1} = \partial_\rho A^*_\gamma - \partial_\gamma A^*_\rho \tag{91}$$

The vector potential is defined as

$$A^*_\gamma = \frac{Q}{4} \int (\kappa_{\rho\lambda} q^\lambda + q^\lambda \kappa^+_{\rho\lambda}) q^*_\gamma \, dx^\rho \tag{92}$$

and can be written as

$$A; \;=\; \$9; \int (\kappa_{\rho\lambda} q^\lambda + q^\lambda \kappa^+_{\rho\lambda}) \, dx^\rho \tag{93}$$

In order to prove that

$$\int 9; \, dx^\rho = 9; \int dx^\rho \tag{94}$$

we can take examples, giving results **such as**

$$\begin{aligned} q^*_Z &= (-q^{(0)}_Z, q^{(1)}_Z, q^{(2)}_Z, q^{(3)}_Z) \\ &= (-q^{(0)}_Z, 0, 0, q^{(3)}_Z) \\ \int q^*_Z \, dX &= q^*_Z \int dX \end{aligned} \tag{95}$$

because q^*_z has no functional dependence on X. The overall structure of the field tensor, using irreducible representations of the Einstein group, is therefore

$$F_{\rho\gamma} = C(\partial_\rho q^*_\gamma - \partial_\gamma q^*_\rho) + D(q_\rho q^*_\gamma - q_\gamma q^*_\rho) \tag{96}$$

where C and D are coefficients. This equation has the structure of a quaternion valued non-Abelian gauge field theory. The most general form of the field tensor

and the vector potential is quaternion-valued. If the following constraint holds

$$\frac{D}{C^2} \equiv -ig \tag{97}$$

the structure of Eq. (96) becomes

$$F_{\rho\gamma} = \partial_\rho A^*_\gamma - \partial_\gamma A^*_\rho - ig[A^*_\rho, A^*_\gamma] \tag{98}$$

which is identical with that of gauge field theory with quaternion-valued potentials. However, the use of the irreducible representations of the Einstein group leads to a structure that is more general than that of Eq. (98). The rules of gauge field theory can be applied to the substructure (98) and to electromagnetism in curved spacetime.

VIII. ANTIGRAVITY EFFECTS IN THE SACHS THEORY

Sachs' equations (4.16) (in Ref. 1)

$$\begin{aligned} \frac{1}{4}(\kappa_{\rho\gamma}q^\lambda + q^\lambda\kappa^+_{\rho\lambda}) + \frac{1}{8}Rq_\rho &= kT_\rho \\ -\frac{1}{4}(\kappa^+_{\rho\gamma}q^{\lambda*} + q^{\lambda*}\kappa_{\rho\gamma}) + \frac{1}{8}Rq^*_\rho &= kT^*_\rho \end{aligned} \tag{99}$$

are 16 equations in 16 unknowns, as these are the 16 components of the quaternion-valued metric. The canonical energy-momentum T_ρ is also quaternion-valued, and the equations are factorizations of the Einstein field equation. If there is no linear momentum and a static electromagnetic field (no Poynting vector), then

$$\mathrm{T}_\rho = (\mathrm{T}^0_\rho, 0, 0, 0) \tag{100}$$

so we have the four components $\mathrm{T}^0_0, \mathrm{T}^0_1, \mathrm{T}^0_2$, and T^0_3. The T^0_0 component is a component of the canonical energy due to the gravitoelectromagnetic field represented by q^0_0. The scalar curvature R is the same with and without electromagnetism, and so is the Einstein constant k.

Considering T^0_0 In Eq. (99), we obtain

$$k\mathrm{T}^0_0 = \frac{1}{8}Rq^0_0 + \frac{1}{4}(\kappa_{0\lambda}q^\lambda + q^\lambda\kappa^+_{0\lambda}) \tag{101}$$

and if we choose a metric such that all components go to zero except q^0_0, then

$$k\mathrm{T}^0_0 \to \frac{1}{8}Rq^0_0 \tag{102}$$

However, R also vanishes in this limit, so

$$\mathrm{T}_0^0 \to 0 \tag{103}$$

So, in order to produce antigravity effects, the gravitoelectromagnetic field must be chosen so that only q_0^0 exists in a static situation. Therefore, antigravity is produced by q_1^0, q_2^0, and q_3^0 all going to zero asymptotically, or by

$$q_0^0 \gg (q_1^0 \approx q_2^0 \approx q_3^0) \tag{104}$$

This result is consistent with the fact that the curvature tensor $\kappa_{0\lambda}$ **must be** minimized, which is a consistent result. The curvature is

$$\kappa_{\rho\lambda} = -\kappa_{\lambda\rho} = \Omega_{\rho;\lambda} - \Omega_{\lambda;\rho} \tag{105}$$

and is minimized if

$$\Omega_{\rho;\lambda} \approx \Omega_{\lambda;\rho} \tag{106}$$

If p = 0, then $\Omega_{0:\lambda} \approx \Omega_{\lambda:0}$. This minimization can occur if the spin–affine connection is minimized. We must now investigate the effect of minimizing $\kappa_{0\lambda}$ on the electromagnetic field

$$F_{\rho\gamma} = Q\left[\frac{1}{4}(\kappa_{\rho\lambda}q^{\lambda}q_{\gamma}^{*} + q_{\gamma}q^{\lambda*}\kappa_{\rho\lambda} + q^{\lambda}\kappa_{\rho\lambda}^{+}q_{\gamma}^{*} + q_{\gamma}\kappa_{\rho\lambda}^{+}q^{\lambda*}) + \frac{1}{8}(q_{\rho}q_{\gamma}^{*} - q_{\gamma}q_{\rho}^{*})R\right] \tag{107}$$

We know that $R \to 0$ and p = 0, so

$$F_{0\gamma} = Q\left[\frac{1}{4}(\kappa_{0\lambda}q^{\lambda}q_{\gamma}^{*} + \cdots)\right] \tag{108}$$

and the $F_{0\gamma}$ component must be minimized. This is the gravitoelectric component. Therefore, the gravitomagnetic component must be very large in comparison with the gravitoelectric component.

IX. SOME NOTES ON QUATERNION-VALUED METRICS

In the flat spacetime limit, the following relation holds:

$$q^{\mu}q^{\nu*} - q^{\nu}q^{\mu*} \to \sigma^{\mu}\sigma^{\nu} - \sigma^{\nu}\sigma^{\mu} \tag{109}$$

where

$$\sigma^{\mu} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \tag{110}$$

Therefore, the quaternion-valued metric can be written as

$$q^{\mu} = \left(\begin{bmatrix} q^{\mu 0} & 0 \\ 0 & q^{\mu 0} \end{bmatrix}, \begin{bmatrix} 0 & q^{\mu 1} \\ q^{\mu 1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -iq^{\mu 2} \\ iq^{\mu 2} & 0 \end{bmatrix}, \begin{bmatrix} q^{\mu 3} & 0 \\ 0 & -q^{\mu 3} \end{bmatrix} \right) \tag{111}$$

with components

$$\begin{aligned}
q^0 &= \left(\begin{bmatrix} q_0^0 & 0 \\ 0 & q_0^0 \end{bmatrix}, \begin{bmatrix} 0 & q_0^1 \\ q_0^1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -iq_0^2 \\ iq_0^2 & 0 \end{bmatrix}, \begin{bmatrix} q_0^3 & 0 \\ 0 & -q_0^3 \end{bmatrix} \right) \\
q_X &= \left(\begin{bmatrix} q_X^0 & 0 \\ 0 & q_X^0 \end{bmatrix}, \begin{bmatrix} 0 & q_X^1 \\ q_X^1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -iq_X^2 \\ iq_X^2 & 0 \end{bmatrix}, \begin{bmatrix} q_X^3 & 0 \\ 0 & -q_X^3 \end{bmatrix} \right) \\
q_Y &= \left(\begin{bmatrix} q_Y^0 & 0 \\ 0 & q_Y^0 \end{bmatrix}, \begin{bmatrix} 0 & q_Y^1 \\ q_Y^1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -iq_Y^2 \\ iq_Y^2 & 0 \end{bmatrix}, \begin{bmatrix} q_Y^3 & 0 \\ 0 & -q_Y^3 \end{bmatrix} \right) \\
q_z &= \left(\begin{bmatrix} q_Z^0 & 0 \\ 0 & q_Z^0 \end{bmatrix}, \begin{bmatrix} 0 & q_Z^1 \\ q_Z^1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -iq_Z^2 \\ iq_Z^2 & 0 \end{bmatrix}, \begin{bmatrix} q_Z^3 & 0 \\ 0 & -q_Z^3 \end{bmatrix} \right)
\end{aligned} \tag{112}$$

In the flat spacetime limit

$$\begin{aligned}
q^0 \to \sigma^0 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 0, 0, 0 \right) \\
q_X \to \sigma_X &= \left(0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 0, 0 \right) \\
q_Y \to \sigma_Y &= \left(0, 0, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, 0 \right) \\
q_Z \to \sigma_Z &= \left(0, 0, 0, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)
\end{aligned} \tag{113}$$

This means that in the flat spacetime limit

$$\begin{array}{llll} q_0^0 \to 1; & q_0^1 \to 0; & q_0^2 \to 0; & q_0^3 \to 0 \\ q_X^0 \to 0; & q_X^1 \to 1; & q_X^2 \to 0; & q_X^3 \to 0 \\ q_Y^0 \to 0; & q_Y^1 \to 0; & q_Y^2 \to 1; & q_Y^3 \to 0 \\ q_Z^0 \to 1; & q_Z^1 \to 0; & q_Z^2 \to 0; & q_Z^3 \to 1 \end{array} \tag{114}$$

Checking with the identity:

$$q_\gamma q^{\kappa *} + q^\kappa q_\gamma^* = 2\sigma_0 \delta_\gamma^\kappa \tag{115}$$

then

$$\begin{aligned} q_X q^{X*} + q^X q_X^* = 2\sigma_0 \delta_X^X = 2\sigma_0 \\ (q_X^0)^2 + (q_X^1)^2 + (q_X^2)^2 + (q_X^3)^2 = \sigma_0 \end{aligned} \tag{116}$$

which is a property of quaternion indices in curved spacetime. In flat spacetime:

$$(q_X^1)^2 = \sigma_0 \tag{117}$$

that is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{118}$$

The reduction to O(3) electrodynamics takes place using products such as

$$\begin{aligned} q_X q_Y^* - q_Y q_X^* &= \begin{bmatrix} 0 & q_X^1 \\ q_X^1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -iq_Y^2 \\ iq_Y^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -iq_Y^2 \\ iq_Y^2 & 0 \end{bmatrix} \begin{bmatrix} 0 & q_X^1 \\ q_X^1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} iq_X^1 q_Y^2 & 0 \\ 0 & -iq_Y^2 q_X^1 \end{bmatrix} \\ &= i\begin{bmatrix} q_Z^3 & 1 \\ 0 & q_Z^3 \end{bmatrix} \end{aligned} \tag{119}$$

that is

$$q_Z^3 = q_X^1 q_Y^2 \tag{120}$$

In flat spacetime, this becomes

$$1 = 1 \tag{121}$$

If the phases are defined as

$$q_X^1 = e^{i\phi}; \quad q_Y^{2*} = e^{-i\phi} \tag{122}$$

then the $\boldsymbol{B}^{(3)}$ field is recovered as

$$B^{(3)} = \frac{1}{8} QR \tag{123}$$

Applying Eq. (99), it is seen that T^μ has the same structure as q^μ:

$$T^\mu = \left(\begin{bmatrix} T^{\mu 0} & o \\ 0 & T^{\mu 0} \end{bmatrix}, \begin{bmatrix} 0 & T^{\mu 1} \\ T^{\mu 1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i\,T^{\mu 2} \\ i\,T^{\mu 2} & 0 \end{bmatrix}, \begin{bmatrix} T^{\mu 3} & 0 \\ 0 & -T^{\mu 3} \end{bmatrix} \right) \tag{124}$$

Therefore, the energy momentum is quaternion-valued. The vacuum current is

$$j_\gamma = \frac{Qk'}{4\pi} (T_\rho^{;\rho} q_\gamma^* - q_\gamma T_\rho^{;\rho *}) \tag{125}$$

where Q and $\kappa'/4\pi$ are constants. We may investigate the structure of the 4-current j_γ by working out the covariant derivative:

$$T_\rho^{;\rho} = \partial^0 T_0 + \partial^1 T_1 + \partial^2 T_2 + \partial^3 T_3 + \Gamma_{0\rho}^\rho T^0 + \Gamma_{1\rho}^\rho T^1 + \Gamma_{2\rho}^\rho T^2 + \Gamma_{3\rho}^\rho T^3 \tag{126}$$

The partial derivatives and Christoffel symbols are not quaternion-valued, so we may write

$$T_\rho^{;\rho} = (\partial^0 + \Gamma_{0\rho}^\rho) T_0 - (\partial^1 + \Gamma_{1\rho}^\rho) T_1 - (\partial^2 + \Gamma_{2\rho}^\rho) T_2 - (\partial^3 + \Gamma_{3\rho}^\rho) T_3 \tag{127}$$

Therefore the vacuum current in general relativity is defined by

$$\begin{aligned} j_\gamma = \frac{Qk'}{4\pi} (((\partial^0 + \Gamma_{0\rho}^\rho) T_0 - (\partial^1 + \Gamma_{1\rho}^\rho) T_1 \quad (\partial^2 + \Gamma_{2\rho}^\rho) T_2 - (\partial^3 + \Gamma_{3\rho}^\rho) T_3) q_\gamma^* \\ + q_\gamma ((\partial^0 + \Gamma_{0\rho}^\rho) T_0 + (\partial^1 + \Gamma_{1\rho}^\rho) T_1 + (\partial^2 + \Gamma_{2\rho}^\rho) T_2 + (\partial^3 + \Gamma_{3\rho}^\rho) T_3)) \end{aligned} \tag{128}$$

This current exists under all conditions and is the most general form of the Lehnert vacuum current described elsewhere in this volume, and the vacuum

current in O(3) electrodynamics. In the Sachs theory, the existence of the electromagnetic field tensor depends on curvature, so energy is extracted from curved spacetime. The 4-current j_μ contains terms such as

$$
\begin{aligned}
j_{\gamma,0} &= \frac{Qk'}{4\pi}((\partial^0 + \Gamma^\rho_{0\rho})T_0 q^*_\gamma + q_\gamma(\partial^0 + \Gamma^\rho_{0\rho})T_0) \\
&= \frac{Qk'}{4\pi}(\partial^0 + \Gamma^\rho_{0\rho})\left(\begin{bmatrix} T^0_0 & 0 \\ 0 & T^0_0 \end{bmatrix} q^*_\gamma + q_\gamma \begin{bmatrix} T^0_0 & 0 \\ 0 & T^0_0 \end{bmatrix}\right)
\end{aligned} \tag{129}
$$

We may now choose $\gamma = 0, 1, 2, 3$ to obtain terms such as

$$
\begin{aligned}
j_{0,0} &= (\partial^0 + \Gamma^\rho_{0\rho})\left(-\begin{bmatrix} T^0_0 & 0 \\ 0 & T^0_0 \end{bmatrix}\begin{bmatrix} q^0_0 & 0 \\ 0 & q^0_0 \end{bmatrix} + \begin{bmatrix} q^0_0 & 0 \\ 0 & q^0_0 \end{bmatrix}\begin{bmatrix} T^0_0 & 0 \\ 0 & T^0_0 \end{bmatrix}\right) = 0 \\
j_{1,0} &= -(\partial^0 + \Gamma^\rho_{0\rho})\left(\begin{bmatrix} T^0_0 & 0 \\ 0 & T^0_0 \end{bmatrix}\begin{bmatrix} 0 & q^1_1 \\ q^1_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & q^1_1 \\ q^1_1 & 0 \end{bmatrix}\begin{bmatrix} T^0_0 & 0 \\ 0 & T^0_0 \end{bmatrix}\right) \\
&= -(\partial^0 + \Gamma^\rho_{0\rho})(q^1_1 T^0_0(\sigma_X + \sigma_0)) \\
&\neq 0
\end{aligned} \tag{130}
$$

There are numerous other components of the 4-current density j_γ that are nonzero under all conditions. These act as sources for the electromagnetic field under all conditions. In flat spacetime, the electromagnetic field vanishes, and so does the 4-current density j_γ.

A check can be made on the interpretation of the quaternion-valued metric if we take the quatemion conjugate:

$$
q^{\mu *} = \left(-\begin{bmatrix} q^{\mu 0} & 0 \\ 0 & q^{\mu 0} \end{bmatrix}, \begin{bmatrix} 0 & q^{\mu 1} \\ q^{\mu 1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & iq^{\mu 2} \\ iq^{\mu 2} & 0 \end{bmatrix}, \begin{bmatrix} q^{\mu 3} & 0 \\ 0 & -q^{\mu 3} \end{bmatrix}\right) \tag{131}
$$

which must reduce, in the f flat space-time limit, to:

$$
\sigma^\mu = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \tag{132}
$$

This means that the flat spacetime metric is

$$
\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -g^{\mu\nu} \tag{133}
$$

which is the negative of the metric $g^{\mu\nu}$ of flat spacetime, that is, Minkowski spacetime.

If we define

$$q^{\mu *}=\left(\begin{bmatrix} q^{\mu 0} & 0 \\ 0 & q^{\mu 0}\end{bmatrix}, -\begin{bmatrix} 0 & q^{\mu 1} \\ q^{\mu 1} & 0\end{bmatrix}, -\begin{bmatrix} 0 & -iq^{\mu 2} \\ iq^{\mu 2} & 0\end{bmatrix}, -\begin{bmatrix} q^{\mu 3} & 0 \\ 0 & -q^{\mu 3}\end{bmatrix}\right) \tag{134}$$

then we obtain

$$g^{\mu\nu} \equiv g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{135}$$

in the flat spacetime limit. This is the usual Minkowski metric

To check on the interpretation given in the text of the reduction of Sachs to O(3) electrodynamics, we can consider generally covariant components such as

$$\begin{aligned} q_X &= (q_X^0, q_X^1, q_X^2, q_X^3) \rightarrow (\sigma^0, \sigma^1, \sigma^2, \sigma^3) \\ q_Y &= (q_Y^0, q_Y^1, q_Y^2, q_Y^3) \rightarrow (\sigma^0, \sigma^1, \sigma^2, \sigma^3) \\ q_Y^* &= (-q_Y^0, q_Y^1, q_Y^2, q_Y^3) \rightarrow (-\sigma^0, \sigma^1, \sigma^2, \sigma^3) \end{aligned} \tag{136}$$

It follows that

$$q_X q_Y^* - q_Y q_X^* \rightarrow \sigma_X \sigma_Y - \sigma_Y \sigma_X = 2i\sigma_Z \tag{137}$$

and that:

$$\begin{aligned} \sigma_X &= (0, \sigma_X, 0, 0) \\ \sigma_Y &= (0, 0, \sigma_Y, 0) \end{aligned} \tag{138}$$

Note that products such as $\sigma_X \sigma_Y$ must be interpreted as single-valued, because products such as

$$[0\ \sigma_x\ 0\ 0]\begin{bmatrix} 0 \\ 0 \\ \sigma_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{139}$$

give a null matrix. Therefore, the quateion-valued product $q_X q_Y^*$ must also be interpreted as

$$q_X q_Y^* - q_Y q_X^* \rightarrow \sigma_X \sigma_Y - \sigma_Y \sigma_X = 2i\sigma_Z \tag{140}$$

as in the text.

Acknowledgments

The U.S. Department of Energy is acknowledged for its Website: http://www.ott.doe.gov/electromagnetic/. This website is reserved for the Advanced Electrodynamics Working Group.

References

1. M. Sachs, I lth chapter in Part 1 of this three-volume set.
2. M. W. Evans, 2nd chapter in this volume (i.e.. Part 2 of this compilation).
3. M. W. Evans, *O(3) Electrodynamics*, Kluwer Academic, Dordrecht, 1999.
4. T. W. Barrett and D. M. Grimes, ***Advanced Electromagnetism***, World Scientific, Singapore, 1995.
5. A demonstration of this property in O(3) electrodynamics is given by Evans and leffers, 1st chapter *in* Parr 3 of this three-volume set; see also Ref(s).—by Vigier in Bibiography at end of that chapter.
6. M. W. Evans, J. P. Vigier. and *S.* Roy (eds.), ***The Enigmatic Photon,*** Kluwer Academic, Dordrecht, 1997. Vo. 4.
7. T. E. Bearden, ***Energy from the Active Vacuum.*** World Scientific, Singapore, in press.
8. T. E. Bearden, I lth and 12th chapters in this volume (i.e., Part 2).
9. M. W. Evans and S. Jeffers, 1 st chapter in Part 3 of thsi three-volume set; see also Ref(s).—by Vigier listed in that chapter.
10. T. W. Barren, in A. Lakhtakia (Ed.), ***Essays on the Formal Aspects Of Electromagnetic Theory,*** World Scientific. Singapore, 1993.
11. L. H. Ryder, Quantum ***Field Theory,*** 2nd ed., Cambridge Univ. Press. Cambridge, UK, 1987.
12. T. W. Barrett, *Apeiron* 7(1), *(2000).*
13. T. E. Bearden et al., *Phys. Scripta* 61, 513 (2000).